SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010

YEAR 12

ASSESSMENT TASK #3

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 120 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise stated.
- Full marks may not be awarded for careless or badly arranged work.

Total Marks – 82

- Attempt questions 1 – 3
- All questions are **NOT** of equal value.
- Each question is to be returned in a separate bundle.

Examiner: *A. Fuller*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Total marks 82

Attempt questions 1 to 3

Answer each **Question** in a **Separate** writing booklet

(Use a SEPARATE writing booklet)

Question 1 (26 marks)

(a) Express $\dfrac{5\pi}{6}$ in degrees. $\boxed{1}$

(b) Find the following correct to 2 decimal places: $\boxed{2}$

(i) $\log_e \frac{3}{2}$

(ii) $\sin 2^c$

(c) Simplify $e^{3\ln x}$ $\boxed{1}$

(d) Differentiate the following with respect to x: $\boxed{5}$

(i) $1 - 2x^2$

(ii) $2\sin x^2$

(iii) e^{1-2x}

(iv) $\dfrac{\cos 2x}{x}$

(v) $(1 - 2\ln x)^2$

(e) State a primitive (indefinite integral) of: $\boxed{5}$

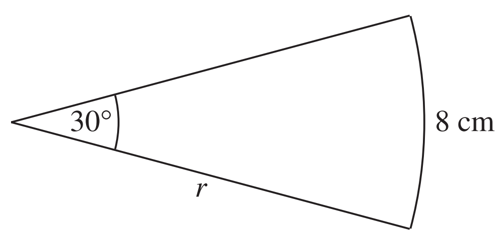
(i) x^{100}

(ii) e^{100x}

(iii) $\sqrt{100x}$

(iv) $\dfrac{100 + x^2}{x^2}$

(f) Find the radius of a sector which has an arc length of 8 cm that subtends an angle of $30°$ at the centre. $\boxed{2}$



(g) (i) Find all the values for x for which $4\cos x + 2 = 0$ where $0 \leq x \leq 2\pi$. $\boxed{4}$

(ii) Hence sketch the graph $y = 4\cos x + 2$ for $0 \leq x \leq 2\pi$ marking clearly where it intersects with the x and y axes.

(h) A function is defined by $f(x) = x^3 - 3x^2 - 12$. $\boxed{6}$

(i) Find the coordinates of the stationary points of the graph $y = f(x)$, and determine their nature.

(ii) Hence sketch the graph of $y = f(x)$.

(iii) From the graph, or otherwise, for what values of x is $y = f(x)$ increasing?

(iv) From the graph, or otherwise, how many real solutions does $x^3 - 3x^2 - 12 = 0$ have?

(Use a SEPARATE writing booklet)

Question 2 (28 marks)

(a) Find the exact value of the following: $\boxed{4}$

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \sin\frac{x}{2}dx$

(ii) $\displaystyle\int_2^6 \frac{dx}{2+x}$

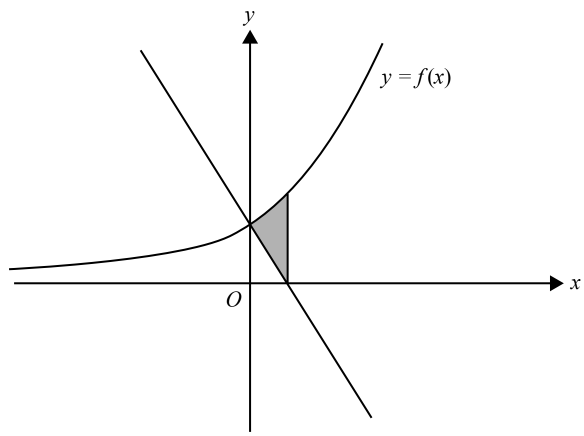
(b) At what point on the curve $y = \ln 2x$ is the gradient of the tangent $\frac{1}{2}$? $\boxed{2}$

(c) For a certain continuous function $f(x)$, $f(2) = 2$ and $f'(2) = -1$. $\boxed{2}$

If $g(x) = x.f(x)$, evaluate $g'(2)$.

(d) Find the volume of the solid of revolution when the area bound by the curve $y = x^2 + 1$, the x-axis, the y-axis and the line $x = 3$ is rotated about the x-axis. $\boxed{3}$

(e) The graph of $y = f(x)$ where $f(x) = e^{\frac{x}{2}} + 1$ is shown below. The normal to the graph of $y = f(x)$ where it crosses the y-axis is also shown. $\boxed{5}$

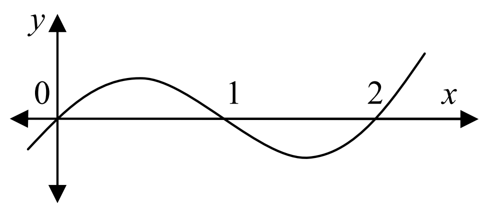


(i) Find the equation of the normal to the graph of $y = f(x)$ where it crosses the y axis.

(ii) Find the exact area of the shaded region.

(f) If $\int_1^3 (2f(x) + 5)dx = 8$ determine the exact value of $\int_1^3 f(x)dx$. $\boxed{2}$

(g) The graph of $y = x(x-1)(x-2)$ is given below. $\boxed{6}$



(i) Expand and simplify $x(x-1)(x-2)$

(ii) Show that $y = x(x-1)(x-2)$ has an inflexion point when $x = 1$.

(iii) Show that $\int_0^2 x(x-1)(x-2)dx = 0$

(iv) $\int_0^1 x(x-1)(x-2)dx = \frac{1}{4}$.

Without evaluating the integral what is the value of $\int_1^2 x(x-1)(x-2)dx$?

(h) A particle moves in a straight line in such a way that its displacement in metres from the origin after t seconds is given by $x = 2t^3 + 3t^2 - 36t + 10$. $\boxed{4}$

(i) In which direction is the particle moving initially?

(ii) When does the particle come to rest?

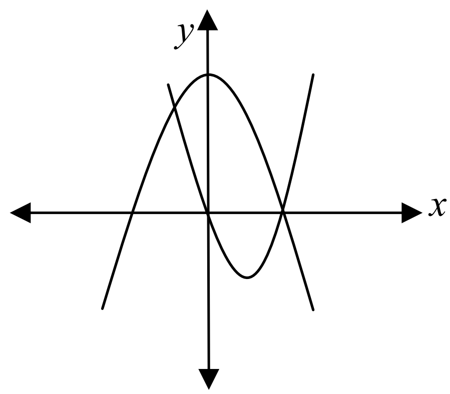
(iii) What is the displacement of the particle after 3 seconds?

(iv) What distance has the particle travelled in the first 3 seconds?

(Use a SEPARATE writing booklet)

Question 3 (28 marks)

(a) The functions $y = 4 - x^2$ and $y = x^2 - 2x$ are sketched below on the same axes. $\boxed{6}$



(i) Copy the above sketch into your answer booklet and label where each function meets the x and y axes.

(ii) Find the points of intersection of the two functions.

(iii) Shade on your diagram in part (i) the region which satisfies the following inequalities: $y \geq x^2 - 2x$, $y \leq 4 - x^2$, $y \geq 0$

(iv) Calculate the area of the shaded region.

(b) (i) Show that $\sec^2 x + \tan^2 x = 2\sec^2 x - 1$ $\boxed{4}$

(ii) By writing $\sec x$ as $(\cos x)^{-1}$ show that $\dfrac{d(\sec x)}{dx} = \sec x \tan x$

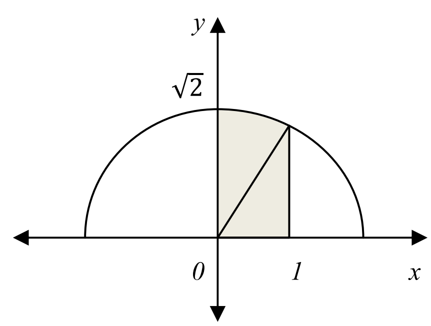
(iii) Hence, or otherwise, find $\displaystyle\int (\sec x + \tan x)^2 dx$

(c) (i) Copy and complete the table in your answer booklet $\boxed{5}$

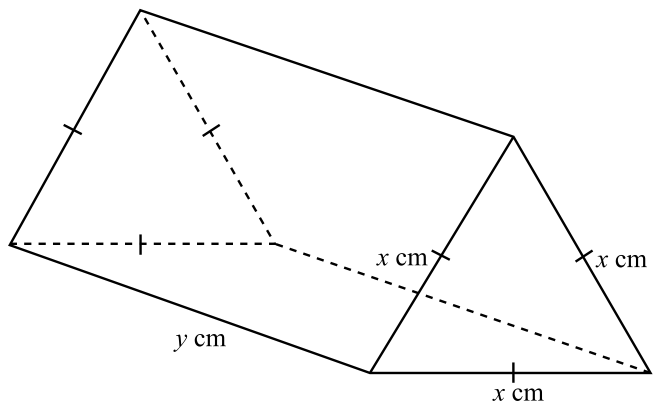
x	0	$\frac{1}{2}$	1
$\sqrt{2-x^2}$			

(ii) Use Simpson's Rule with three function values to approximate $\displaystyle\int_0^1 \sqrt{2-x^2}dx$ to 2 decimal places.

(iii) By considering the area below, find the exact value of $\displaystyle\int_0^1 \sqrt{2-x^2}dx$



(d) A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm. $\boxed{6}$



The volume of the brick is 1000 cm^3.

(i) Show that the area of the equilateral triangle is given by $\dfrac{\sqrt{3}x^2}{4}$.

(ii) Find an expression for y in terms of x.

(iii) Show that the total surface area, A cm^2, of the brick is given by

$$A = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}.$$

(iv) Find the value of x for which the brick has minimum total surface area.

(e) $A(5, 20)$, $B(30, 15)$, $C(20, -10)$ and D are the vertices of a quadrilateral $ABCD$. $\boxed{7}$

Given that the diagonals AC and BD are perpendicular.

(i) Prove that the point D lies on the line $y = \frac{x}{2}$.

(ii) If also $AB = AD$, prove that the coordinates of D are $(-6, -3)$.

(iii) Prove that AC bisects BD.

(iv) What type of quadrilateral is $ABCD$?

End of paper

Mathematics

Assessment 3, 2010

Question 1:

a. $\frac{5\pi}{6} \times \frac{180}{\pi} = 150^\circ$ (1)

b. i. $\log_e(3/2) = 0.4405465108$

$= 0.41$ (2dp) (1)

ii. $\sin 2^c = 0.9092974268$

$= 0.91$ (2dp) (1)

c. $e^{3\ln x} = (e^{\ln x})^3$

$= x^3$ (1)

d. i. $\frac{d}{dx}(1 - 2x^2) = -4x$ (1)

ii. $\frac{d}{dx}(2\sin x^2) = 2\cos x^2 \times 2x$

$= 4x\cos x^2$ (1)

iii. $\frac{d}{dx}(e^{1-2x}) = -2(e^{1-2x})$

$= -2e^{1-2x}$ (1)

iv. $\frac{d}{dx}\left(\frac{\cos 2x}{x}\right)$ $- u$ $- v$

$\frac{d}{dx} = \frac{vu' - uv'}{v^2}$

$u = \cos 2x \qquad v = x$

$u' = -2\sin 2x \qquad v' = 1$

$= \frac{-2x\sin 2x - \cos 2x}{x^2}$ (1)

v. $\frac{d}{dx}(1 - 2\ln x)^2$

$= 2(1 - 2\ln x) \times -\frac{2}{x}$

$= -\frac{4}{x}(1 - 2\ln x)$ (1)

e. i. $\int x^{100}\,dx$

$= \frac{x^{101}}{101} + C$ (1)

-1 for first instance of no +C only

ii. $\int e^{100x}\,dx = \frac{1}{100}e^{100x} + C$ (1)

iii. $\int \sqrt{100x}\,dx = \int \sqrt{100}\cdot\sqrt{x}\,dx$

$= \int 10x^{1/2}\,dx$

$= \frac{10x^{3/2}}{3/2} + C$

$= \frac{20x^{3/2}}{3} + C$ (1)

iv. $\int \frac{100 + x^2}{x^2}\,dx = \int \frac{100}{x^2} + \frac{x^2}{x^2}\,dx$

$= \int (100x^{-2} + 1)\,dx$

$= \frac{100x^{-1}}{-1} + x + C$

$= x - \frac{100}{x} + C$ (2)

f. $l = r\theta$

$\theta = 30^\circ = \frac{\pi}{6}^c$

$8 = r \times \frac{\pi}{6}$ (2)

$\therefore r = \frac{48}{\pi}$

g.i. $4\cos x + 2 = 0$

$0 \le x \le 2\pi$

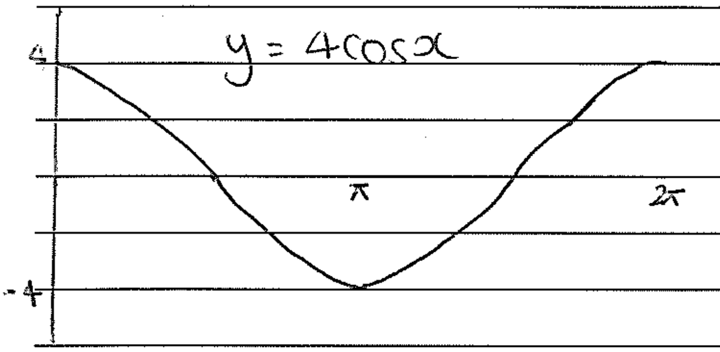
$4\cos x = -2$

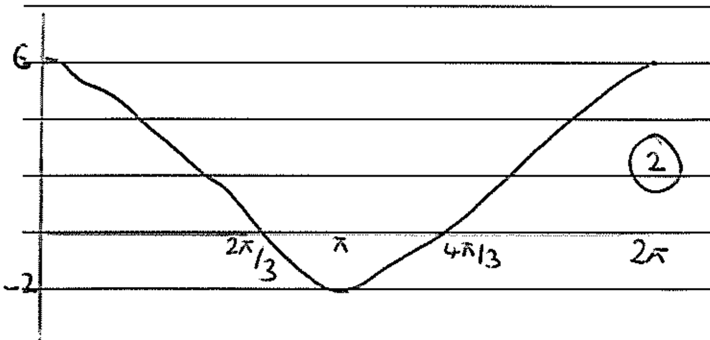
$\cos x = -\frac{1}{2}$

$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}$

S	A
T	C

(2)





(2)

h.i. $f(x) = x^3 - 3x^2 - 12$

$f'(x) = 3x^2 - 6x$

for stat. points $f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore x = 0, 2$

$f(0) = 0 - 0 - 12 = -12$

$f(2) = 2^3 - 3(2)^2 - 12 = -16$

$\therefore$ stat points are $(0, -12)$ & $(2, -16)$ (1)

$(0, -12)$

x	-1	0	1
$f'(x)$	7	0	-3

/ — \ $\therefore$ max.

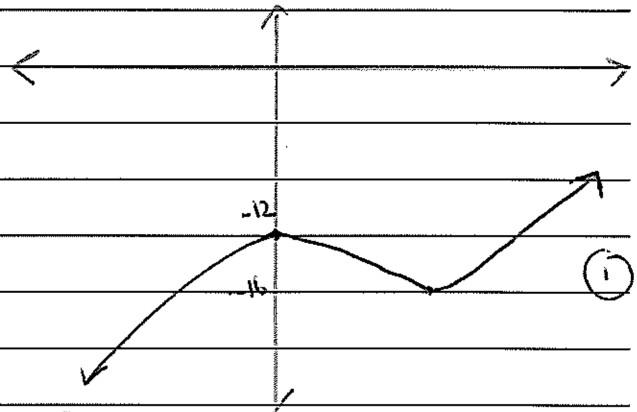
$(2, -16)$

x	1	2	3
$f'(x)$	-3	0	9

(1)

\ — / $\therefore$ min

ii.



(1)

iii. $f(x)$ is increasing for $x < 0$, $x > 2$ (2)

iv. 1 solution (1)

Q2 (a)(i) $\int_0^{\pi/2} \sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{\pi/2}$

$= -2\left[\cos\frac{\pi}{4} - \cos 0\right]$

$= -2\left[\frac{1}{\sqrt{2}} - 1\right]$

$= 2 - \sqrt{2}$.

(ii) $\int_2^6 \frac{dx}{2+x} = \left[\ln(x+2)\right]_2^6$

$= \ln 8 - \ln 4$

$= \ln\frac{8}{4}$

$= \ln 2$

(b) $y = \ln 2x$

$y' = 2 \cdot \frac{1}{2x} = \frac{1}{x}$

when $y' = \frac{1}{2}$, $\frac{1}{x} = \frac{1}{2} \longrightarrow x = 2$

$\therefore$ Point is $(2, \ln 4)$

(c) $f(2) = 2 \qquad f'(2) = -1$

$g(x) = x * f(x)$

$\therefore$ By Product Rule $g'(x) = f(x) \times 1 + x \times f'(x)$

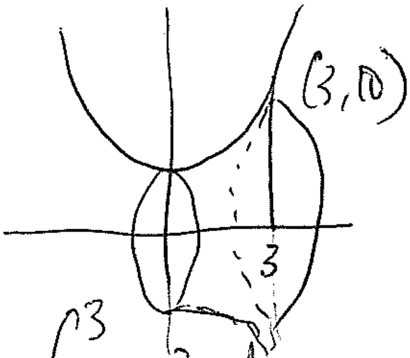
$\therefore g'(2) = f(2) + 2f'(2)$

$= 2 + 2 \times -1$

$= 0$.

(d) $y = x^2 + 1$

When $x = 3$, $y = 10$



$\therefore$ Required volume $= \pi \int_0^3 y^2\, dx$

$= \pi \int_0^3 \left[x^2 + 1\right]^2 dx$

$= \pi \int_0^3 \left(x^4 + 2x^2 + 1\right)$

$= \pi \left[\frac{x^5}{5} + \frac{2}{3}x^3 + x\right]_0^3$

$= \pi \left\{\left[\frac{243}{5} + \frac{2}{3}\cdot 27 + 3\right] - \left[0\right]\right\}$

$= \pi \left\{48\frac{3}{5} + 18 + 3\right\}$

$= 69\frac{3}{5}\pi$ cubic units

(e) (i) $y = e^{\frac{x}{2}} + 1$

$y' = \frac{1}{2} e^{\frac{x}{2}}$

At $x = 0$, $y' = \frac{1}{2} e^0 = \frac{1}{2}$.

$\therefore m_{TANGENT} = \frac{1}{2} \longrightarrow m_{NORMAL} = \frac{-2}{1} = -2$

At $x = 0$, $y = e^0 + 1 = 2$

$\therefore$ Equation of normal is $y = -2x + 2$

ie $2x + y - 2 = 0$.

(ii) Required area $= \int_0^1 (e^{\frac{x}{2}} + 1)\,dx - \int_0^1 (-2x + 2)\,dx$

$= \int_0^1 (e^{\frac{x}{2}} + 1 + 2x - 2)\,dx$

$= \int_0^1 (e^{\frac{x}{2}} + 2x - 1)\,dx$

$= \left[2e^{\frac{x}{2}} + x^2 - x\right]_0^1$

$= \left[2e^{\frac{1}{2}} + 1^2 - 1\right] - \left[2e^0 + 0 - 0\right]$

$= \left(2e^{\frac{1}{2}} - 2\right)$ sq. units

(1) $\int_1^3 [2f(x)+5]\,dx = 8$

$\therefore \int_1^3 2f(x)\,dx + \int_1^3 5\,dx = 8$

ie $2\int_1^3 f(x)\,dx + \left[5x\right]_1^3 = 8$

$2\int_1^3 f(x)\,dx + [15-5] = 8$

ie $2\int_1^3 f(x)\,dx = 8-10$

$= -2$

ie $\int_1^3 f(x)\,dx = -1.$

(9) $y = x(x-1)(x-2)$

(i) $y = x(x^2-3x+2)$

$= x^3 - 3x^2 + 2x$

(ii) $y' = 3x^2 - 6x + 2$

$y'' = 6x - 6$

$y'' = 0 \rightarrow 6x - 6 = 0 \rightarrow x = 1.$

$\therefore$ Possible point of inflexion at $x=1$.

Check for change of sign of $f''(x)$ at $x=1$.

x	0	1	2
	$-$	0	$+$

ie There is a change of sign

$\therefore$ $f(x)$ has point of inflexion at $x = 1$

(iii) $\int_0^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^2$

$= \left[\frac{16}{4} - 8 + 4\right] - [0]$

$= 4 - 8 + 4$

$= 0$

(iv) Now $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

But $\int_0^2 f(x)\,dx = 0$ [from iii] and $\int_0^1 f(x)\,dx = \frac{1}{4}$ [given]

$\therefore \int_1^2 f(x)\,dx$ must be $-\frac{1}{4}$.

(4) $x = 2t^3 + 3t^2 - 36t + 10$

(i) $x' = 6t^2 + 6t - 36$

when $t = 0$, $x' = -36$ (ie x' is negative)

$\therefore$ Particle is moving to the left.

(ii) $x' = 0 \to 6t^2 + 6t - 36 = 0$

ie $t^2 + t - 6 = 0$

$\therefore (t+3)(t-2) = 0$

ie $t = -3, 2$

But $t \geq 0$, $\therefore$ Particle comes to rest when $t = 2$

(iii) When $t = 3$, $x = 54 + 27 - 108 + 10$

$= 91 - 108$

$= -17$

(iv) When $t = 0$, $x = 10$ metres

when it stops after 2 secs, $x = 16 + 12 - 72 + 10$

$= 38 - 72$

$= -34$ metres

Particle has travelled 44 units in 1st 2 secs

When $t = 3$, $x = -17$ metres

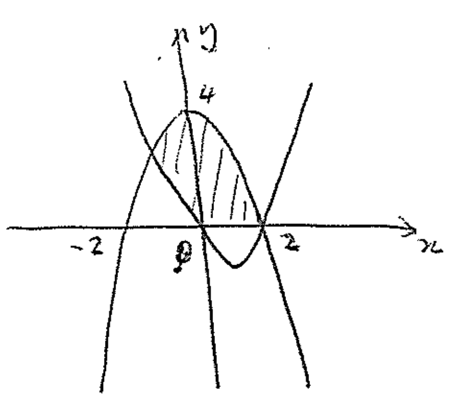
ie Particle has moved from -34 to -17 in the 3rd sec

$\therefore$ Particle has travelled 17 metres in 3rd second

$\therefore$ Distance travelled in 1st 3 secs $= 44 + 17 = 61$ m

3 (a)

(i)



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(ii) $4 - x^2 = x^2 - 2x$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$ 2

Point of intersection $(2,0)$, $(-1,3)$

(iv) Area $= \int_{-1}^{0} \left((4-x^2) - (x^2 - 2x)\right) dx + \int_{0}^{2} (4 - x^2)\, dx$

$= \int_{-1}^{0} (4 + 2x - 2x^2)\, dx + \int_{0}^{2} (4 - x^2)\, dx$

$= \left[4x + x^2 - \frac{2}{3}x^3\right]_{-1}^{0} + \left[4x - \frac{1}{3}x^3\right]_{0}^{2}$

$= [0] - [-4 + 1 + \frac{2}{3}] + [8 - \frac{8}{3}] - [0]$ 2

$= 2\frac{1}{3} + 5\frac{1}{3}$

$= 7\frac{2}{3}$ units^2

6

b) (i) $\sec^2 x + \tan^2 x = \sec^2 x + \sec^2 x - 1$

$= 2\sec^2 x - 1$ 1

(ii) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left((\cos x)^{-1}\right)$

$= -1(\cos x)^{-2} \cdot -\sin x$

$= \dfrac{\sin x}{\cos^2 x}$ 1

$= \tan x \cdot \sec x$.

(iii) $\int (\sec x + \tan x)^2\, dx$

$= \int (\sec^2 x + 2\sec x \tan x + \tan^2 x)\, dx$

$= \int (2\sec^2 x + 2\sec x \tan x - 1)\, dx$ 2

$= 2\tan x + 2\sec x - x + c$. 4

(c) (i)

x	0	$\frac{1}{2}$	1
$\sqrt{2-x^2}$	$\sqrt{2}$	$\frac{\sqrt{7}}{2}$	1

1

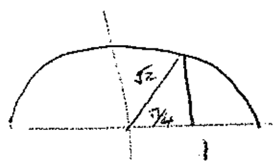
(ii) $\int_0^1 \sqrt{2-x^2}\, dx = \dfrac{\frac{1}{2}}{3}\left(\sqrt{2} + 4\cdot\frac{\sqrt{7}}{2} + 1\right)$

$= \frac{1}{6}\left(\sqrt{2} + 2\sqrt{7} + 1\right)$ 2

$= ~~0.979719~~ \ldots$ $1.284286\ldots$

$\approx$ ~~0.98~~ 1.28

(iii) $\int_0^1 \sqrt{2-x^2}\, dx = \frac{1}{2}\cdot(\sqrt{2})^2\cdot\frac{\pi}{4} + \frac{1}{2}\times 1\times 1$



$= \frac{\pi}{4} + \frac{1}{2}$

$(\doteq 1.28539\ldots)$ 2

5

(d) (i) Area of $\Delta = \frac{1}{2} x^2 . \sin \frac{\pi}{3}$

$$= \frac{\sqrt{3}x^2}{4}$$

(ii) Volume $= 1000 = \frac{\sqrt{3}x^2}{4} \times y$.

$$\therefore y = \frac{4000}{\sqrt{3}x^2}$$

(iii) $A = 3 \times \frac{4000}{\sqrt{3}x^2} \times x + 2 \times \frac{\sqrt{3}x^2}{4}$

$$= \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}$$

(iv) For min A $\frac{d(A)}{dx} = 0$.

$\frac{dA}{dx} = -4000\sqrt{3}x^{-2} + \sqrt{3}x \therefore 4000\sqrt{3} \times -1 x^{-2} + \sqrt{3}x = 0$.

$$\therefore \sqrt{3}x^3 - 4000\sqrt{3} = 0$$

$$\therefore x^3 = 4000$$

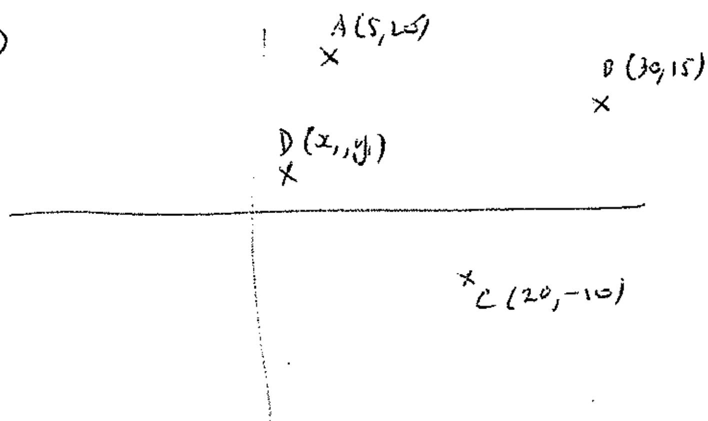
$$\therefore x = 10\sqrt[3]{4}$$

$\frac{d^2A}{dx^2} = +8000\sqrt{3}x^{-3} + \sqrt{3}$.

when $x = 10\sqrt[3]{4}$, $\frac{d^2A}{dx^2} > 0$ 3

$\therefore$ Min Surface area when $x = 10\sqrt[3]{4}$ (6)

(e)



(i) $m_{AC} = \frac{30}{-15} = -2$

$m_{BD} = \frac{15-y_1}{30-x_1}$

$m_{AC} \times m_{BD} = -1$.

$\therefore -2 \times \frac{15-y_1}{30-x_1} = -1$

$\therefore \frac{15-y_1}{30-x_1} = \frac{1}{2}$

$\therefore 15 - y_1 = 15 - \frac{1}{2}x_1$

$\therefore y_1 = \frac{1}{2}x_1$ 2

$\therefore$ D lies on $y = \frac{1}{2}x$.

(ii) $AB = AD$

$\sqrt{(30-5)^2+(15-20)^2} = \sqrt{(x_1-5)^2+(y_1-20)^2}$

$\therefore \sqrt{625+25} = \sqrt{x_1^2-10x_1+25+y_1^2-40y_1+400}$

$\therefore 650 = x_1^2-10x_1+25+\frac{1}{4}x_1^2-20x_1+400$

$\therefore 225 = \frac{5}{4}x_1^2 - 30x_1$

$\therefore 900 = 5x_1^2 - 120x_1$

+1 -180 $\therefore 180 = x_1^2 - 24x_1$

2 -90 $x_1^2 - 24x_1 - 180 = 0$

3 -60 $(x_1+6)(x_1-30) = 0$ 2

4 -45 $\therefore x_1 = -6$ or $x_1 = 30$

5 -36 $y_1 = -3$ $y_1 = 15$

6 -30

$\therefore$ D is $(-6, -3)$

(iii) $M_{BD} = (12, 6) = M$

$$m_{AM} = \frac{20-6}{5-12} = -2 = m_{AC}$$

2

$\therefore$ M lies on AC

$\therefore$ AC bisects BD

(iv) ABCD is a kite

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